2019 年度日本政府（文部科学省）奨学金留学生選考試験

QUALIFYING EXAMINATION FOR APPLICANTS FOR THE JAPANESE GOVERNMENT (MEXT) SCHOLARSHIP 2019

学科試験　　問題

EXAMINATION QUESTIONS

（学部留学生）

UNDERGRADUATE STUDENTS

数　　学　（A）

MATHEMATICS(A)

注意　☆試験時間は **60 分**。

PLEASE NOTE: THE TEST PERIOD IS **60 MINUTES**.

Nationality		No.	
Name	(Please print full name, underlining family name)		

Marks	

1. Answer the following questions in the corresponding boxes on the answer sheet.

(1) Let a point P move on a straight line according to the score shown on a fair dice that we throw by the following rules. P starts from the origin O.

- If the score is 6, then P returns to the origin O.
- If the score is 1, 2, or 3, then P moves 1 in a positive direction.
- If the score is 4 or 5, then P moves 1 in a negative direction.

When we throw the dice four times, the probability that the point P is at the origin O is $\boxed{\text{[1-1]}}$.

(2) For a constant k, we consider the number of distinct real solutions of equation $x|x^2-3x+2| = k$. The range of k that the number of real solutions is maximum is $\boxed{\text{[1-2]}} < k < \boxed{\text{[1-3]}}$, and the maximum number of real solutions is $\boxed{\text{[1-4]}}$.

(3) Assume that $0 < \theta < \pi$. For three points $\mathrm{A}(1, 0)$, $\mathrm{B}(\cos\theta, \sin\theta)$, and $\mathrm{C}(\cos 2\theta, \sin 2\theta)$ on a unit circle, the area of $\triangle\mathrm{ABC}$ is $\boxed{\text{[1-5]}}$ by using θ. When $\theta = \boxed{\text{[1-6]}}$, the maximum of the area of $\triangle\mathrm{ABC}$ is $\boxed{\text{[1-7]}}$.

(4) Let k be a positive integer and let p be a prime number that is greater than 2. The sum of all divisors of the number $2^k p$ is

$$\left(\boxed{\text{[1-8]}} - 1\right)\left(1 + \boxed{\text{[1-9]}}\right),$$

where all divisors include 1 and the number itself.

(5) In a box, there are 10 cards and a number from 1 to 10 is written on each card. When three cards from the box are randomly taken at a time, we define X, Y, and Z according to three numbers in ascending order. The probability that X is less than or equal to 3 is $\boxed{\text{[1-10]}}$.

(6) The n-th term of sequence $1, 4, 10, 19, 31, \ldots$ is $\boxed{\text{[1-11]}}$, and the sum of the first n terms of the sequence is $\boxed{\text{[1-12]}}$.

(7) Let a and b be positive real numbers.

$$\frac{4a+b}{2a} + \frac{4a-3b}{b}$$

is at minimum when $b = \boxed{\text{[1-13]}}\, a$. Its minimum value is $\boxed{\text{[1-14]}}$.

(8) For a variable x, we have

$$(x+1)^n = \sum_{k=0}^{n} {}_nC_k \boxed{\text{[1-15]}}^{\boxed{\text{[1-16]}}}.$$

It follows that

$$\sum_{k=0}^{n} {}_nC_k 2^k = \boxed{\text{[1-17]}}^{\boxed{\text{[1-18]}}}.$$

By considering the derivatives of the first equality in this item with respect to x, we have

$$\sum_{k=0}^{n} {}_nC_k k 2^k = \frac{\boxed{\text{[1-19]}}}{\boxed{\text{[1-20]}}} \sum_{k=0}^{n} {}_nC_k 2^k.$$

(9) For a positive integer n, let x_k ($k = 0, 1, \ldots, n$) be an integer between 0 and 5. We have

$$\sum_{k=0}^{n} x_k 6^k = \boxed{\text{[1-21]}} + \boxed{\text{[1-22]}} \left(\sum_{k=1}^{n} x_k \sum_{l=0}^{k-1} 6^l \right)$$

so that a senary (base 6) number can be divided by $\boxed{\text{[1-22]}}$ with no remainder if and only if the sum of all of its digits can be divided by $\boxed{\text{[1-23]}}$ with no remainder.

(10) It is clear that $253x + 256y = 253(x + y) + 3y$. For a pair of integers x and y satisfying

$$253x + 256y = 1,$$

the absolute value of x is minimum. Then, $x = \boxed{\text{[1-24]}}$ and $y = \boxed{\text{[1-25]}}$.

(11) Translate the graph of the function $y = 2x^2 + 3x + 1$ by 2 units in the x-direction and by -3 units in the y-direction and express the resulting graph by

$$y = a_2 x^2 + a_1 x + a_0.$$

Then, we have $a_2 = \boxed{\text{[1-26]}}$, $a_1 = \boxed{\text{[1-27]}}$, $a_0 = \boxed{\text{[1-28]}}$.

2. For a triangle ABC, take a point D on side AB such that side CD is orthogonal to side AB. We let $\angle \mathrm{BAC} = \frac{\pi}{12}$ and let the lengths of side AB and side AD be $2\sqrt{2}$ and $\sqrt{6}$, respectively. Answer the following questions in the corresponding boxes on the answer sheet. They should be simplified as much as possible.

(1) From $\pi/12 = \pi/3 - \pi/4$, we have

$$\cos\frac{\pi}{12} = \frac{\boxed{[2\text{-}1]} + \sqrt{2}}{4}.$$

(2) The length of side AC is

$$\boxed{[2\text{-}2]} - 2\sqrt{3}.$$

(3) The square of the length of side BC, $(\mathrm{BC})^2$, is

$$\boxed{[2\text{-}3]} - 32\sqrt{3}.$$

(4) Thus, the length of side BC is

$$\boxed{[2\text{-}4]} - 2\sqrt{6}.$$

3. For a quadratic function $f(x)$, we define a function as follows:

$$F(x) = \int_0^x f(t)\,dt.$$

Assume that a is a positive number and the function $F(x)$ has extreme values at $x = -2a, 2a$. Answer the following questions in the corresponding boxes on the answer sheet.

(1) For any x, it holds that

$$F(-x) = \boxed{\text{[3-1]}}\,F(x).$$

(2) All the values of x that satisfy $F(x) + F(2a) = 0$ are $\boxed{\text{[3-2]}}$.

(3) The local maximum value of function $\dfrac{F(x)}{F'(0)}$ is $\boxed{\text{[3-3]}}$.